

Arrangement of hyperplanes I: Rational functions and Jeffrey-Kirwan residue

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1 Introduction

Consider the space R_Δ of rational functions of r variables with poles on an arrangement of hyperplanes Δ . It is important to study the decomposition of the space R_Δ under the action of the ring of differential operators with constant coefficients. In the one variable case, a rational function of z with poles at most on $z = 0$ is written uniquely as $\phi(z) = \text{Princ}(\phi)(z) + \psi(z)$ where $\text{Princ}(\phi)(z) = \sum_{n < 0} a_n z^n$ is the principal part of $\phi(z)$ and $\psi(z) = \sum_{n \geq 0} a_n z^n$ is the polynomial part of $\phi(z)$. Remark that the space

$$G = \{\phi(z) = \sum_{n < 0} a_n z^n\}$$

of principal parts is free under the action of $\partial/\partial z$ while the space of polynomials is evidently a torsion module. Furthermore, the function $1/z$ is the unique function which cannot be written as a derivative.

We show similarly, in the case of several variables, that there is a well determined decomposition of R_Δ as

$$R_\Delta = G_\Delta \oplus NG_\Delta$$

where G_Δ is a free module under the action of the ring of differential operators with constant coefficients, and NG_Δ is the torsion submodule. Here the space G_Δ can be characterized as the space of rational functions with a zero at infinity in all directions. Let us describe more precisely the space G_Δ . We need more notations.

Let V be a finite dimensional vector space over a field k , of characteristic zero. Let $r = \dim V$. Let Δ be a finite subset of nonzero elements of V .

Consider the union of hyperplanes in V^* :

$$\mathcal{H}^*(\Delta) := \bigcup_{\alpha \in \Delta} \{z \in V^*, \langle z, \alpha \rangle = 0\}$$

and the ring R_Δ of rational functions on V^* with poles contained in $\mathcal{H}^*(\Delta)$. We denote by G_Δ the subspace of R_Δ spanned by the elements

$$\frac{1}{\prod_{\alpha \in \kappa} \alpha^{n_\alpha}}$$

where κ is a subset of Δ generating V and where the n_α are positive integers. It turns out that G_Δ is the subspace of R_Δ consisting of functions that vanish at infinity in any direction. It is a graded vector space with highest graded part $G_\Delta[-r] := S_\Delta$. Furthermore, S_Δ is the linear span of the

$$\phi_\sigma = \frac{1}{\prod_{\alpha \in \sigma} \alpha}$$

where σ ranges over all bases of Δ .

As the space G_Δ is a direct factor in R_Δ , under the action of the ring $S(V^*)$ of differential operators with constant coefficients, there is a natural projection Res_Δ from R_Δ to S_Δ that we call the Jeffrey-Kirwan residue. The name Residue is justified by the fact that the kernel of the map Res_Δ is the space of derivatives, and by a generalization of the Cauchy formula. Any $S(V^*)$ -morphism from G_Δ to another $S(V^*)$ -module is entirely determined by its value on S_Δ , and this morphism exists provided certain linear relations between the ϕ_σ are satisfied. The space S_Δ is isomorphic to the top degree component of the “Orlik-Solomon algebra” associated to the hyperplane arrangement $\mathcal{H}^*(\Delta)$; as a consequence, we produce bases of S_Δ consisting of certain ϕ_σ . Their dual bases can be described in terms of iterated residues, as shown by Szenes (see [7] and section 4).

If $k = \mathbb{R}$, then G_Δ occurs as the space of Laplace transforms of locally polynomial functions with possible discontinuities on hyperplanes generated by $r - 1$ elements of Δ . The Laplace transform intertwines the action of $S(V^*)$ on locally polynomial functions by multiplication, with its action on G_Δ by differential operators with constant coefficients. We study the jumps of locally polynomial functions in terms of the poles of their Laplace transforms. As a consequence, we show that a locally polynomial function is continuous if and only if its Laplace transform vanishes at order 2 in any direction. We

also construct inverses of the Laplace transform, using our description of G_Δ by generators and relations.

Many of the statements proved in this article are already implicitly stated in Jeffrey-Kirwan articles [4] and [5]. However, we felt the need, for applications, to clarify some of their statements. The main application will be an algebraic construction of Eisenstein series: to each rational function with poles on hyperplanes, we will associate a periodic meromorphic function in several variables. This will be treated in part II of this article.

Applications to the Poisson formula will be given in another article.

Our interest in the space of functions R_Δ and their Laplace transforms comes from the study of integrals over symplectic spaces of equivariant cohomology classes. Let (M, Ω) be a compact symplectic manifold, with an Hamiltonian action of a torus T . Let $f : M \rightarrow \mathfrak{t}^*$ be the moment map. Let $X \in \mathfrak{t}$. Let $\Omega(X) = \langle f, X \rangle + \Omega$ be the equivariant symplectic form. Let $\alpha(X)$ be an equivariant closed form on M . Consider the integral

$$I(X) = \int_M \alpha(X) e^{\Omega(X)}.$$

Assume for simplicity that the set F of fixed points for the action of T on M is finite. For $p \in F$, let $\Delta_p \subset \mathfrak{t}^*$ be the set of weights for the action of T in the tangent space $T_p M$. Then, by the localisation formula in equivariant cohomology, we have

$$I(X) = \sum_{p \in F} \phi_p(X) e^{\langle f(p), X \rangle}$$

where each ϕ_p is in the ring R_{Δ_p} .

If ξ is a regular value of f , we can consider the reduced space $M_{red}(\xi) = f^{-1}(\xi)/T$ with reduced symplectic structure Ω_ξ . The equivariant cohomology class of $\alpha(X)$ gives rise to a de Rham cohomology class α_ξ on $M_{red}(\xi)$. Consider the function

$$r(\xi) = \int_{M_{red}(\xi)} \alpha_\xi e^{\Omega_\xi}.$$

This function is defined for regular values of ξ . It is important to determine this function and its jumps when crossing walls of singular values of the moment map. The functions $I(X)$ and $r(\xi)$ are related by the Laplace transform. Thus it is important to study jumps of Laplace transforms of functions in the space G_Δ .

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2 Rational functions with poles on hyperplanes: Jeffrey-Kirwan residue

Let V be a finite dimensional vector space over a field k , of characteristic zero. Let $r = \dim V$. We denote by $S(V)$ the symmetric algebra of V . Let V^* be the dual space. We identify $S(V)$ with the ring of polynomial functions on V^* . Let $\Delta \subset V$ be a finite subset of nonzero elements, which spans V . We denote by

$$R_\Delta := \Delta^{-1}S(V)$$

the ring generated over $S(V)$ by inverting the linear functions $\alpha \in \Delta$. This is a ring graded by the degree (positive or negative). Consider the union of hyperplanes in V^* :

$$\mathcal{H}^*(\Delta) := \bigcup_{\alpha \in \Delta} \{z \in V^*, \langle \alpha, z \rangle = 0\}$$

and the open subset

$$V_{reg, \Delta}^* = V^* - \mathcal{H}^*(\Delta)$$

of (Δ) -regular elements in V^* . Then R_Δ is the ring of rational functions on V^* with poles contained in the union of hyperplanes $\mathcal{H}^*(\Delta)$. Functions in R_Δ are defined on the set $V_{reg, \Delta}^*$ of regular elements.

Let \mathcal{D} be the ring of differential operators on V^* with polynomial coefficients. Recall that the ring \mathcal{D} is generated by its subrings $S(V)$ of polynomial functions on V^* , and $S(V^*)$ of differential operators on V^* with constant coefficients. Observe that $S(V)$ and R_Δ are graded \mathcal{D} -modules.

If $\phi \in R_\Delta$ and if $y \in V^*$ is a regular element, then $t \mapsto \phi(y + tz)$ is a rational function for any $z \in V^*$. We say that ϕ **vanishes at infinity** if the rational function $t \mapsto \phi(y + tz)$ is 0 at ∞ for all regular $y \in V^*$ and for all $z \in V^*$.

Let κ be a subset of Δ . The subset κ is called **generating** if the $\alpha \in \kappa$ generate the vector space V . It is called a **basis** of Δ , if the $\alpha \in \kappa$ form a basis of V . We denote by $\mathcal{B}(\Delta)$ the set of bases of Δ .

For $\kappa \subset \Delta$, set

$$\phi_\kappa := \frac{1}{\prod_{\alpha \in \kappa} \alpha}.$$

We denote by G_Δ the subspace of R_Δ spanned by the

$$\frac{1}{\prod_{\alpha \in \kappa} \alpha^{n_\alpha}}$$

where κ is **generating** and the n_α are positive integers. Then G_Δ is a graded vector space with highest graded part S_Δ (in degree $-r$). Furthermore, S_Δ is the linear span of the ϕ_σ where σ ranges over all bases of Δ .

Clearly, any function in the space G_Δ vanishes at infinity. We will prove that the converse holds in Theorem 1 below.

Remark. The space G_Δ is contained in $\sum_{j \geq r} R_\Delta[-j]$ but is strictly smaller if $r > 1$. For example, if $\alpha \in \Delta$, then α^{-r} is never in G_Δ .

We denote by NG_Δ the subspace of R_Δ spanned by the

$$\frac{\psi}{\prod_{\alpha \in \kappa} \alpha^{n_\alpha}}$$

where $\psi \in S(V)$, κ is **not generating** and the n_α are non-negative integers.

Remark that the subspace NG_Δ of R_Δ is stable under the action of \mathcal{D} , whereas G_Δ is stable under the action of $S(V^*)$ by differential operators with constant coefficients.

Theorem 1 *We have a direct sum decomposition of $S(V^*)$ -modules*

$$R_\Delta = G_\Delta \oplus NG_\Delta.$$

Moreover, the space G_Δ is a free $S(V^)$ -module, and is freely generated by S_Δ , while the space NG_Δ is the torsion submodule. Finally, G_Δ is the space of functions in R_Δ which vanish at infinity.*

For this we prove a succession of lemmas.

Lemma 2 *The $S(V^*)$ -module G_Δ is generated by S_Δ . Moreover, we have $R_\Delta = G_\Delta + NG_\Delta$.*

Proof. Observe that the $S(V^*)$ -module generated by S_Δ is the span of the elements

$$\frac{1}{\prod_{\beta \in \sigma} \beta^{n_\beta}}$$

where $\sigma \in \mathcal{B}(\Delta)$ and where each n_β is a positive integer. To prove the first assertion, it is enough to check that this vector space is stable by multiplication by $1/\alpha^n$ where $\alpha \in \Delta$. For this, write $\alpha = \sum_{\beta \in \sigma} c_{\alpha\beta} \beta$. Then we have

$$\frac{1}{\alpha^n \prod_{\beta \in \sigma} \beta^{n_\beta}} = \sum_{\beta \in \sigma} \frac{c_{\alpha\beta}}{\alpha^{n+1} \beta^{n_\beta-1} \prod_{\gamma \in \sigma, \gamma \neq \beta} \gamma^{n_\gamma}}.$$

If $\beta \in \sigma$ is such that $n_\beta = 1$, then the corresponding term in the right-hand side is in the $S(V^*)$ -module generated by $G_\Delta[-r]$: indeed, if $c_{\alpha\beta} \neq 0$ then $\sigma \cup \{\alpha\} \setminus \{\beta\}$ is a basis of Δ . On the other hand, if $n_\beta > 1$ then our term is the inverse of $\alpha^{n+1} \prod_{\beta \in \sigma} \beta^{n'_\beta}$ with $n'_\beta \geq 1$ and $\sum_{\beta \in \sigma} n'_\beta = (\sum_{\beta \in \sigma} n_\beta) - 1$. So the assertion follows by induction on $\sum_{\beta \in \sigma} n_\beta$.

Similarly, any element of $R_\Delta = \Delta^{-1}S(V)$ is a linear combination of elements

$$\phi = \frac{\psi}{\prod_{\alpha \in \kappa} \alpha^{n_\alpha}}$$

where $\psi \in S(V)$, κ is linearly independent and the n_α are positive integers. If moreover κ is not generating, then ϕ is in NG_Δ . If κ is generating, then we can express ψ as a polynomial in the variables $\alpha \in \kappa$, and we obtain $\phi \in G_\Delta + NG_\Delta$.

Lemma 3 *The $S(V^*)$ -module R_Δ/NG_Δ is free.*

Proof. Observe that R_Δ/NG_Δ is a \mathcal{D} -module. Furthermore, it is spanned (as a vector space) by the images of

$$\frac{1}{\prod_{\alpha \in \sigma} \alpha^{n_\alpha}}$$

where σ is a basis of Δ , and the n_α are positive integers. It follows that the \mathcal{D} -module R_Δ/NG_Δ is generated by the images $\overline{\phi_\sigma}$ of the ϕ_σ ($\sigma \in \mathcal{B}(\Delta)$). Observe that $\overline{\phi_\sigma}$ is killed by V ; thus, the \mathcal{D} -module $\mathcal{D}\overline{\phi_\sigma}$ is a non zero quotient of $\mathcal{D}/\mathcal{D}V$. The latter is a simple \mathcal{D} -module, isomorphic to $S(V^*)$; therefore, $\mathcal{D}\overline{\phi_\sigma}$ is isomorphic to $S(V^*)$, too. Iterating this argument, we construct an ascending filtration of the \mathcal{D} -module R_Δ/NG_Δ , each submodule being generated by certain $\overline{\phi_\sigma}$'s, with successive quotients isomorphic to $S(V^*)$.

Lemma 4 *The subspace S_Δ intersects NG_Δ trivially.*

Proof. We argue by induction on the number of elements in Δ . We may assume that Δ contains no proportional elements. Let $\phi \in S_\Delta$. Write

$$\phi = \sum_{\sigma \in \mathcal{B}(\Delta)} \frac{c_\sigma}{\prod_{\alpha \in \sigma} \alpha}$$

and consider ϕ as a rational function on V^* . Observe that the poles of ϕ are simple and along the hyperplanes $\alpha = 0$ ($\alpha \in \Delta$). Choose α among the poles of ϕ . Choose a decomposition $V = k\alpha \oplus V_0$. Then $Q(V)$ (the fraction field of $S(V)$) is identified with the field of rational functions in the variable α , with coefficients in $Q(V_0)$. Therefore, we have a restriction map $S(V) \rightarrow S(V_0) : \phi \mapsto \phi_0$. Consider the image Δ_0 of $\Delta \setminus \{\alpha\}$ in V_0 . The restriction map extends to an homomorphism $(\Delta \setminus \{\alpha\})^{-1}S(V) \rightarrow \Delta_0^{-1}S(V_0)$ by restriction to generic points. We have also a residue map $Res_\alpha : Q(V) \rightarrow Q(V_0)$ with respect to the variable α , defined by the formula

$$Res_\alpha(\phi) = \frac{1}{(K-1)!} \left(\left(\frac{\partial}{\partial \alpha} \right)^{K-1} (\alpha^K \phi) \right)_0$$

for any integer K such that $\alpha^K \phi \in R_{\Delta \setminus \{\alpha\}}$.

As α is a simple pole of ϕ , we have simply

$$Res_\alpha(\phi) = \sum_{\sigma, \alpha \in \sigma} \frac{c_\sigma}{\prod_{\beta \in \sigma, \beta \neq \alpha} \beta_0}$$

where β_0 denotes the image of β in V_0 . If σ is a basis of Δ which contains α , then $(\sigma \setminus \{\alpha\})_0$ is a basis of Δ_0 . Therefore, $Res_\alpha(\phi)$ is in G_{Δ_0} .

Consider a generator

$$u = \frac{\psi}{\prod_{\beta \in \kappa} \beta^{n_\beta}}$$

of NG_Δ , with $\psi \in S(V)$ and κ non generating. Write

$$u = \frac{\psi}{\alpha^K \prod_{\beta \in \kappa, \beta \neq \alpha} \beta^{n_\beta}}.$$

If $K = 0$, then $Res_\alpha(u) = 0$. If $K > 0$, the set κ contains α and is non generating. Thus, its restriction κ_0 is non generating. We see that $Res_\alpha(u)$ can be written as

$$Res_\alpha(u) = \frac{\psi'}{\prod_{\beta_0 \in \kappa_0} \beta_0^{n_{\beta_0} + K - 1}}.$$

for some $\psi' \in S(V_0)$, so that $Res_\alpha(u) \in NG_{\Delta_0}$.

If $\phi \in G_\Delta \cap NG_\Delta$, it follows from the above discussion that $Res_\alpha(\phi) \in G_{\Delta_0} \cap NG_{\Delta_0}$. Therefore, by the induction hypothesis, we have $Res_\alpha(\phi) = 0$: thus, ϕ has no pole along $\alpha = 0$. By the beginning of the proof, ϕ has no pole at all, so that $\phi = 0$.

Lemma 5 *If $\phi \in R_\Delta$ vanishes at infinity, then ϕ is in G_Δ .*

Proof. First we claim that the space of functions which vanish at infinity is stable by the action of $S(V^*)$. Indeed, let $\phi \in R_\Delta$ vanish at infinity. Write

$$\phi = \frac{\psi}{\prod_{\alpha \in \Delta} \alpha^{n_\alpha}}$$

where $\psi \in S(V)$. For $z \in V^*$, set

$$n(z) := \sum_{\alpha, \langle \alpha, z \rangle \neq 0} n_\alpha.$$

The assumption that ϕ vanishes at infinity means that

$$\deg(t \mapsto \psi(y + tz)) < n(z)$$

for all regular y and for all z in V^* . Let $w \in V^*$; then, for all $u \in k$ such that $y + uw$ is regular, we also have

$$\deg(t \mapsto \psi(y + tz + uw)) < n(z)$$

and therefore, the function

$$\frac{\partial(w)\psi}{\prod_{\alpha \in \Delta} \alpha^{n_\alpha}}$$

vanishes at infinity. Now

$$\partial(w)\phi = \frac{\partial(w)\psi}{\prod_{\alpha \in \Delta} \alpha^{n_\alpha}} - \sum_{\alpha \in \kappa} \frac{n_\alpha \langle \alpha, w \rangle}{\Delta} \phi$$

which implies our claim.

Assume now that there exists a non-zero $\phi \in NG_\Delta$ which vanishes at infinity. As in the proof of Lemma 2, we can write

$$\phi = \sum_{\kappa} \phi_\kappa$$

where the sum is over all linearly independent subsets $\kappa \subset \Delta$ which are not bases, and where each ϕ_κ is in $\kappa^{-1}S(V)$. Furthermore, we may assume that the number of κ such that ϕ_κ is non-zero is minimal (among all possible decompositions of all non-zero $\phi \in NG_\Delta$ which vanish at infinity).

Choose κ_0 such that $\phi_{\kappa_0} \neq 0$, and choose a non-zero $z_0 \in V^*$ such that $\langle \alpha, z_0 \rangle = 0$ for all $\alpha \in \kappa_0$. Then $\partial^n(z_0)\phi_{\kappa_0} = 0$ for large n . But all successive derivatives of ϕ vanish at infinity and are in NG_Δ . Moreover,

$$\partial^n(z_0)\phi = \sum_{\kappa \neq \kappa_0} \partial^n(z_0)\phi_\kappa$$

is a decomposition with fewer terms than ϕ . Thus, $\partial^n(z_0)\phi = 0$ for some positive n .

Choose n minimal with this property, and set $\psi := \partial^{n-1}(z_0)\phi$. Then ψ is a non-zero element of NG_Δ which vanishes at infinity, and $\partial(z_0)\psi = 0$. But then the function $t \mapsto \psi(y + tz_0)$ is constant for any $y \in V^*$, a contradiction.

The space S_Δ is generated by the elements ϕ_σ where σ ranges over $\mathcal{B}(\Delta)$. However, there are linear relations among the elements ϕ_σ . Indeed, let σ be a basis of Δ . If $\alpha \in \Delta \setminus \sigma$ and

$$\alpha = \sum_{\beta \in \sigma} c_{\alpha\beta} \beta$$

is the expansion of α in the basis σ , then $\sigma \cup \{\alpha\} \setminus \{\beta\}$ is a basis if and only if $c_{\alpha\beta}$ is non zero, and we have

$$\phi_\sigma = \sum_{\beta \in \sigma, c_{\alpha\beta} \neq 0} c_{\alpha\beta} \phi_{\sigma \cup \{\alpha\} \setminus \{\beta\}}.$$

In section 4, we will prove that the linear relations between the elements ϕ_σ are generated by the relations above.

We can now define the Jeffrey-Kirwan residue map: denote by

$$\hat{R}_\Delta := \Delta^{-1} \hat{S}(V)$$

the ring generated over the ring $\hat{S}(V)$ of formal power series, by inverting the linear functions $\alpha \in \Delta$. Define the Taylor expansion at order K as the projection

$$Taylor_{[\leq K]} : \hat{R}_\Delta \rightarrow \bigoplus_{j \leq K} R_\Delta[j].$$

Using $Taylor_{[\leq -r]}$, we project the space \hat{R}_Δ to $R_\Delta[\leq -r]$. Then using the direct sum decomposition

$$R_\Delta = G_\Delta \oplus NG_\Delta$$

we obtain a projection map

$$Princ_{\Delta} : \hat{R}_{\Delta} \rightarrow G_{\Delta}$$

by composing both projections $\hat{R}_{\Delta} \rightarrow R_{\Delta}[\leq -r] \rightarrow G_{\Delta}$.

Remark that as G_{Δ} is contained in $R_{\Delta}[\leq -r]$, the map $Princ_{\Delta}$ can also be defined as the composition of $Taylor_{[\leq K]} : \hat{R}_{\Delta} \rightarrow R_{\Delta}[\leq K]$ for any index $K \geq -r$, followed by the projection $R_{\Delta}[\leq -K] \rightarrow G_{\Delta}$.

Definition 6 *The Jeffrey-Kirwan residue map*

$$Res_{\Delta} : \hat{R}_{\Delta} \rightarrow S_{\Delta}$$

is defined to be the composite of the projection $Princ_{\Delta}$ followed by the projection of G_{Δ} on S_{Δ} .

In other words, the map Res_{Δ} is the identity on S_{Δ} and vanishes on $\oplus_{j \neq -r} R_{\Delta}[j]$ and on NG_{Δ} as well. We can determine easily the map Res_{Δ} on \hat{R}_{Δ} by first projecting on $R_{\Delta}[-r]$, then, using the fact that Res_{Δ} vanishes on NG_{Δ} , projecting further on $G_{\Delta}[-r] = S_{\Delta}$.

Consider the subspace V^*R_{Δ} spanned by derivatives of elements of R_{Δ} ; it is a submodule of R_{Δ} under the action of $S(V^*)$.

Proposition 7 *We have*

$$V^*R_{\Delta} = NG_{\Delta} \oplus \bigoplus_{j < -r} G_{\Delta}[j].$$

In particular, we have

$$R_{\Delta} = V^*R_{\Delta} \oplus S_{\Delta}, \quad \hat{R}_{\Delta} = V^*\hat{R}_{\Delta} \oplus S_{\Delta}$$

and the kernel of Res_{Δ} is $V^*\hat{R}_{\Delta}$.

Proof. From Theorem 1 we obtain

$$V^*R_{\Delta} = V^*NG_{\Delta} \oplus V^*G_{\Delta} = V^*NG_{\Delta} \oplus \bigoplus_{j < -r} G_{\Delta}[j].$$

So it is sufficient to check that $NG_{\Delta} = V^*NG_{\Delta}$. For this, consider

$$\phi = \frac{\psi}{\prod_{\alpha \in \kappa} \alpha^{n_{\alpha}}}$$

where $\psi \in S(V)$ and where κ is linearly dependent. Choose $y \in V^*$ such that $\langle y, \alpha \rangle = 0$ for all $\alpha \in \kappa$. We can find $\Psi \in S(V)$ such that $\partial(y)\Psi = \psi$; then

$$\phi = \partial(y)\left(\frac{\Psi}{\prod_{\alpha \in \kappa} \alpha^{n_\alpha}}\right).$$

In particular, the kernel of Res_Δ is the space of derivatives. Using Res_Δ , we now obtain a multidimensional analogue of the Cauchy formula: for any meromorphic function ϕ of one variable z , and for any $y \neq 0$, we have

$$(Princ \phi)(y) = Res_{z=0} \frac{\phi(z)}{y - z}.$$

Let $y \in V^*$ be regular and let $\psi \in R_\Delta$. Set

$$(C(y)\psi)(z) := \psi(y - z).$$

Then the rational function $C(y)\psi$ is defined at 0, and thus its Taylor series at the origin is in $\hat{S}(V)$. To any $\phi \in \hat{R}_\Delta$, we associate the endomorphism $u(\phi)$ of S_Δ defined by

$$(u(\phi)\psi)(z) = Res_\Delta(\phi(z)\psi(y - z)).$$

Denoting by $m(\phi)$ the multiplication by ϕ , then $u(\phi)$ is composition $Res_\Delta \circ m(\phi) \circ C(y)$. We consider its trace $Tr_{S_\Delta}(Res_\Delta \circ m(\phi) \circ C(y))$.

Proposition 8 *For any regular y in V^* and for any $\phi \in \hat{R}_\Delta$, we have*

$$(Princ_\Delta \phi)(y) = Tr_{S_\Delta}(Res_\Delta \circ m(\phi) \circ C(y)).$$

Proof. First we consider the case where $\phi \in S_\Delta$. Then

$$Res_\Delta(\phi(z)\psi(y - z)) = \phi(z)\psi(y)$$

because $z \mapsto \psi(y - z)$ is defined at 0. So $u(\phi)$ maps ψ to $\psi(y)\phi$, and its trace is $\phi(y) = Princ_\Delta(\phi)(y)$.

Now we assume that the formula holds for ϕ , and we claim that it holds for $\partial(w)\phi$ where $w \in V^*$. Indeed, using the fact that Res_Δ vanishes on derivatives, we obtain

$$Res_\Delta((\partial(w)\phi)(z)\psi(y - z)) = -Res_\Delta(\phi(z)\partial_z(w)\psi(y - z))$$

$$= \text{Res}_\Delta(\phi(z)\partial_y(w)\psi(y-z)) = \partial_y(w)\text{Res}_\Delta(\phi(z)\psi(y-z))$$

which implies the claim.

It follows that the formula holds for any $\phi \in G_\Delta$. If $\phi \in \widehat{NG_\Delta}$ then the left-hand side vanishes. On the other hand, the function $z \mapsto \psi(y-z)$ is in $\hat{S}(V)$; thus, $z \mapsto \phi(z)\psi(y-z)$ is in $\widehat{NG_\Delta}$ and the right-hand side vanishes, too.

Remark. More generally, let $A : \hat{R}_\Delta \rightarrow \hat{R}_\Delta$ be an operator which commutes with the action of $S(V^*)$. Then we have for any regular $y \in V^*$ and for any $\phi \in \hat{R}_\Delta$:

$$A(\text{Princ}_\Delta(\phi))(y) = \text{Tr}_{S_\Delta}(\text{Res}_\Delta \circ m(\phi) \circ C(y) \circ A)$$

(the proof is the same).

Let us deduce from this (abstract) Cauchy formula, an explicit expression of $\text{Princ}_\Delta(\phi)$ in terms of derivatives of elements of S_Δ . For this, choose a basis $(\phi_b)_{b \in B}$ of S_Δ and denote by (ϕ^b) the dual basis. For $\phi \in \hat{R}_\Delta$ and $h \in V$, the function $y \mapsto e^{-\langle y, h \rangle} \phi(y)$ is in \hat{R}_Δ . Moreover, the map

$$h \mapsto \langle \phi^b, \text{Res}_\Delta(e^{-h} \phi) \rangle := D^b(\phi)(h)$$

is easily seen to be polynomial. It thus defines a differential operator $D^b(\phi)$ on V^* .

Proposition 9 *For any $\phi \in \hat{R}_\Delta$, and for any basis $(\phi_b)_{b \in B}$ of S_Δ , we have*

$$\text{Princ}_\Delta(\phi) = \sum_{b \in B} D^b(\phi) \cdot \phi_b.$$

Proof. Let y be a regular element of V^* . Then we have by the Cauchy formula:

$$\begin{aligned} \text{Princ}_\Delta(\phi)(y) &= \text{Tr}_{S_\Delta}(\text{Res}_\Delta \circ m(\phi) \circ C(y)) \\ &= \sum_{b \in B} \langle \phi^b, \text{Res}_\Delta(\phi(z)\phi_b(y-z)) \rangle. \end{aligned}$$

Now observe that $\phi_b(y-z) = (e^{-\partial(z)} \phi_b)(y)$. Thus, we have

$$\langle \phi^b, \text{Res}_\Delta(\phi(z)\phi_b(y-z)) \rangle = D^b(\phi) \cdot \phi_b.$$

Remark that Propositions 13 and 14 below provide a basis $(\phi_b)_{b \in B}$ together with the dual basis $(\phi^b)_{b \in B}$. Thus we obtain an explicit expression of any element in G_Δ as a sum of successive derivatives of elements ϕ_σ . This provides a way of separating variables.

Example. Let V be a vector space with basis (e_1, e_2) . Let Δ be the ordered set

$$\Delta = (e_1, e_2, e_1 + e_2).$$

The set B of Proposition 13 according to this ordering consists of

$$b_1 = (e_1, e_2) \quad b_2 = (e_1, e_1 + e_2).$$

Furthermore, if $\sigma = \{e_2, e_1 + e_2\}$, we have $\phi_\sigma = \phi_{b_1} - \phi_{b_2}$. Let

$$\phi(z_1, z_2) = \frac{1}{z_1 z_2 (z_1 + z_2)}.$$

If $h = h_1 e_1 + h_2 e_2$, the component of degree -2 of $e^{-h_1 z_1 - h_2 z_2} \phi(z_1, z_2)$ is

$$\frac{-h_1 z_1 - h_2 z_2}{z_1 z_2 (z_1 + z_2)} = -\frac{h_1}{z_2 (z_1 + z_2)} - \frac{h_2}{z_1 (z_1 + z_2)} = -h_1 \phi_{b_1} + (h_1 - h_2) \phi_{b_2}.$$

We have indeed

$$\frac{1}{z_1 z_2 (z_1 + z_2)} = -\frac{\partial}{\partial z_1} \cdot \frac{1}{z_1 z_2} + \left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) \cdot \frac{1}{z_1 (z_1 + z_2)}.$$

Remark. The residue that Jeffrey and Kirwan actually defined is a linear form over S_Δ , defined in the case when $k = \mathbb{R}$. It depends on choices of chambers in V and V^* . We will describe this residue in section 5.

3 Residue along a hyperplane

Let us recall the notion of a residue map along a hyperplane.

Let V_0 be an hyperplane in V . We denote by Δ_0 the subset $\Delta \cap V_0$. The space V_0^\perp is a line in V^* . The fibers of the restriction map $V^* \rightarrow V_0^*$ are affine lines $z + V_0^\perp$. If ϕ is a rational function with poles on the set of hyperplanes Δ , its restriction to the affine line $z + V_0^\perp$ is a rational function, except when the affine line $z + V_0^\perp$ is contained in the pole set of ϕ (in this case the restriction

is nowhere defined). The residue at infinity of this rational function is well defined. More precisely, choose differential forms of maximal degree ω on V^* , ω_0 on V_0^* and choose an equation z_0 of V_0 , such that $\omega_0 = \text{int}(z_0)\omega$ where int is the contraction. Define the residue map

$$\text{Res}_{V/V_0} : \Delta^{-1}S(V) \otimes \wedge^r V \rightarrow \Delta_0^{-1}S(V_0) \otimes \wedge^{r-1}V_0$$

by

$$\text{Res}_{V/V_0}(\phi \otimes \omega)(z) = -\text{Res}_{t=\infty}(\phi(z + tz_0)dt) \otimes \omega_0$$

for $z \in V^*$ (clearly, this only depends on the image of z in $V^*/kz_0 = V_0^*$). We now give a characterization of this map.

We identify $R_{\Delta_0} = \Delta_0^{-1}S(V_0)$ to a subalgebra of R_Δ , so that R_Δ is a R_{Δ_0} -module. We denote by Δ_1 the complement of Δ_0 in Δ .

If $\nu = (\alpha_j, 1 \leq j \leq L)$ is a sequence of elements of Δ with possible repetitions, we set

$$m_\nu := \frac{1}{\prod_{j=1}^L \alpha_j}.$$

We write $\nu \subset \Delta_0$ (resp. $\nu \subset \Delta_1$) if all elements α_j of the sequence ν are in Δ_0 (resp. Δ_1).

Let $\omega \in \wedge^r V$. If $\beta \in \Delta_1$, there exists a unique $\omega_0 \in \wedge^{r-1}V_0$ such that $\omega = \beta \wedge \omega_0$. We then write

$$\omega_0 = (d\beta)^{-1} \wedge \omega.$$

Proposition 10 *The map Res_{V/V_0} is the unique R_{Δ_0} -linear map*

$$\text{res}_{V/V_0} : R_\Delta \otimes \wedge^r V \rightarrow R_{\Delta_0} \otimes \wedge^{r-1}V_0$$

such that, for $\omega \in \wedge^r V$,

1) for any $\beta \in \Delta_1$,

$$\text{res}_{V/V_0}(\frac{1}{\beta} \otimes \omega) = (d\beta)^{-1} \wedge \omega.$$

2)

$$\text{res}_{V/V_0}(S(V) \otimes \omega) = 0.$$

3) for any sequence $\nu \subset \Delta_1$:

$$\text{res}_{V/V_0}(m_\nu \otimes \omega) = 0$$

if the length of ν is strictly greater than 1.

Indeed, these properties are easily checked for the map Res_{V/V_0} defined above, and uniqueness follows from the following remark.

Proposition 11 *We have*

$$R_\Delta = \Delta_0^{-1}S(V) + \sum_{\nu \subset \Delta_1} R_{\Delta_0} m_\nu.$$

Proof. Let $\psi \in S(V)$ and ν a sequence of elements of Δ . Consider the element ψm_ν of $R_\Delta = \Delta^{-1}S(V)$. If ν is contained in Δ_0 , or if $\psi \in S(V_0)$, we are already in the desired set. If $\alpha_j \in \nu$ is not in Δ_0 and if ψ is not in $S(V_0)$, then using the decomposition

$$S(V) = S(V_0) \oplus \alpha_j S(V)$$

we can strictly decrease the power of α_j in the expression of m_ν .

We finally note some properties of the map Res_{V/V_0} .

We extend the map $Princ_\Delta : R_\Delta \rightarrow G_\Delta$ to a map

$$Princ_\Delta : R_\Delta \otimes \wedge^{max} V \rightarrow G_\Delta \otimes \wedge^{max} V$$

still denoted by $Princ_\Delta$. In the same way we extend the map Res_Δ to a map

$$Res_\Delta : R_\Delta \otimes \wedge^{max} V \rightarrow S_\Delta \otimes \wedge^{max} V.$$

Proposition 12 *The map Res_{V/V_0} is homogeneous of degree -1 , and is compatible with the maps $Princ$ and with the Jeffrey-Kirwan residue. More explicitly:*

1)

$$Res_{V/V_0}(Princ_\Delta(\phi)) = Princ_{\Delta_0}(Res_{V/V_0}(\phi)).$$

2)

$$Res_{V/V_0}(Res_\Delta(\phi)) = Res_{\Delta_0}(Res_{V/V_0}(\phi)).$$

Proof. Remark that

$$NG_\Delta \subset \Delta_0^{-1}S(V) + \sum_{\nu \subset \Delta_1} NG_{\Delta_0} m_\nu.$$

Therefore, Res_{V/V_0} maps NG_Δ to NG_{Δ_0} , and both members of equation (1) vanish on NG_Δ .

Now consider an element $m_\nu = m_{\nu_0} m_{\nu_1}$ where ν is generating. If the length of ν_1 is greater than 1, both members of equation (1) vanish. If ν_1 consists of one element, then ν_0 generates V_0 and we obtain Assertion 1. Assertion 2 follows from the fact that Res_{V/V_0} is homogeneous of degree -1 .

4 Orlik-Solomon relations

In this section, we describe the linear relations between the generators ϕ_σ ($\sigma \in \mathcal{B}(\Delta)$) of the space S_Δ , and we construct bases of this space consisting of certain ϕ_σ . Using iterated residues, we construct the dual bases as well.

For this, we begin by interpreting the space S_Δ in terms of the **Orlik-Solomon algebra** associated to the hyperplane arrangement $\mathcal{H}^*(\Delta)$, see [6] Chapter 3. Recall that this algebra, which we denote by A_Δ , is the subalgebra of rational differential forms on V^* generated by the forms

$$\omega_\alpha := \frac{d\alpha}{\alpha}$$

where $\alpha \in \Delta$. Clearly, A_Δ is graded by the degree of differential forms, and its top degree component is

$$A_\Delta[r] = S_\Delta \otimes \wedge^r V.$$

It is known that the algebra A_Δ is the quotient of the free exterior algebra on symbols e_α ($\alpha \in \Delta$) by its ideal generated by the elements

$$\sum_{j=1}^s (-1)^{j-1} e_{\alpha_1} \wedge \cdots \wedge \widehat{e_{\alpha_j}} \wedge \cdots \wedge e_{\alpha_s}$$

where $\alpha_1, \dots, \alpha_s \in \Delta$ are linearly dependent (see [6] 3.5). It follows that the space $A_\Delta[r]$ is generated by the elements

$$\omega_{(\alpha_1, \dots, \alpha_r)} := \omega_{\alpha_1} \wedge \cdots \wedge \omega_{\alpha_r} = \frac{d\alpha_1 \wedge \cdots \wedge d\alpha_r}{\alpha_1 \cdots \alpha_r}$$

where $(\alpha_1, \dots, \alpha_r)$ is an ordered basis of Δ . Moreover, the linear relations between the $\omega_{(\alpha_1, \dots, \alpha_r)}$ are consequences of the relations

$$\sum_{1 \leq j \leq r, c_j \neq 0} (-1)^{j-1} \omega_{\alpha_1, \dots, \widehat{\alpha_j}, \dots, \alpha_r, \alpha} = 0$$

where $(\alpha_1, \dots, \alpha_r)$ is as above, and where $\alpha = \sum_{j=1}^r c_j \alpha_j$.

Finally, a basis of $A_\Delta[r]$ consisting of certain $\omega_{(\alpha_1, \dots, \alpha_r)}$ can be defined as follows (see [2], [6] 3.2 and [7]). Choose an ordering $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N)$ of Δ . Consider the subset $B \subset \mathcal{B}(\Delta)$ consisting of the ordered bases $b =$

$(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n})$ (listed with strictly increasing indices) such that for all $j \neq i_p$, the set $\{\alpha_j\} \cup \{\alpha_{i_p}; i_p > j\}$ is linearly independent. Then the ω_b ($b \in B$) are the desired basis.

Translating these results in terms of S_Δ leads to the following

Proposition 13 *The set $(\phi_b)_{b \in B}$ is a basis of S_Δ . Furthermore, the space of linear relations between the ϕ_σ ($\sigma \in \mathcal{B}(\Delta)$) is generated by the “Orlik-Solomon relations”*

$$r_{\sigma, \alpha} := \phi_\sigma - \sum_{\beta \in \sigma} c_{\alpha\beta} \phi_{\sigma \cup \{\alpha\} \setminus \{\beta\}}$$

where $\sigma \in \mathcal{B}(\Delta)$, $\alpha \in \Delta \setminus \sigma$ and $\alpha = \sum_{\beta \in \sigma} c_{\alpha\beta} \beta$.

For completeness, we will present an a priori proof of this result; first, let us give an example.

Example. Let V be a vector space with basis e_1, e_2, e_3 . Consider the ordered set

$$\Delta = (e_1, e_2, e_3, e_1 + e_2, e_2 + e_3, e_1 + e_2 + e_3)$$

(the set of positive roots of a root system of type A_3). Then the set B consists of

$$b_1 = (e_1, e_2, e_3), \quad b_2 = (e_1, e_3, e_1 + e_2), \quad b_3 = (e_1, e_2, e_2 + e_3),$$

$$b_4 = (e_1, e_1 + e_2, e_2 + e_3), \quad b_5 = (e_1, e_2, e_1 + e_2 + e_3), \quad b_6 = (e_1, e_3, e_1 + e_2 + e_3).$$

Proof. Let L be the free vector space with basis the elements ϕ_σ , $\sigma \in \mathcal{B}(\Delta)$. Let LR be the kernel of the natural map from L to R_Δ . By definition, LR is the space of linear relations between the ϕ_σ . We denote by C the subspace of L with basis $(\phi_b)_{b \in B}$, and by OSR the subspace of LR generated by the elements $r_{\sigma, \alpha}$. Let us show that $L = C + OSR$. If $\sigma = (\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r})$ is an ordered basis of Δ , we set $|\sigma| = i_1 + i_2 + \dots + i_r$. If σ is not in B , then there exists a j such that the set $\{\alpha_j\} \cup \{\alpha_{i_p}; i_p > j\}$ is linearly dependent. Using the relation r_{σ, α_j} , we replace ϕ_σ by a linear combination of elements ϕ_τ where τ is obtained from σ by replacing one of the elements α_{i_p} with $i_p > j$ by α_j . It follows that the numbers $|\tau|$ are strictly smaller than $|\sigma|$, so that by induction, we obtain $L = C + OSR$. This shows that the set $(\phi_b)_{b \in B}$ generates S_Δ , and that LR is spanned by $C \cap LR$ and by OSR .

We now show that $C \cap LR = 0$. We need to check that if $\sum_{b \in B} c_b \phi_b = 0$ as a rational function, then all c_b are equal to 0. We prove this by induction on

the number of elements in Δ . Remark that all elements of B contain α_1 . For a set κ of $r-1$ linearly independent vectors, let $H(\kappa) \subset V$ be the hyperplane generated by κ . For a hyperplane $H \subset V$, set $\mathcal{B}(H) := \{\kappa \subset \Delta; H(\kappa) = H\}$. We write

$$\sum_{b \in B} c_b \phi_b = \alpha_1^{-1} \sum_H \phi_H$$

with

$$\phi_H := \sum_{b, H(b \setminus \{\alpha_1\}) = H} \frac{c_b}{\prod_{\beta \in b \setminus \{\alpha_1\}} \beta}.$$

Choose a hyperplane H_0 generated by $b \setminus \{\alpha_1\}$, for some $b \in B$. Then the residue operator Res_{V/H_0} kills all elements $\alpha_1^{-1} \phi_H$ except $\alpha_1^{-1} \phi_{H_0}$, which is mapped to ϕ_{H_0} . Thus, $\phi_{H_0} = 0$. But remark that if we consider the ordered set $\Delta_0 = \Delta \cap H_0$, the set $B_0 = B(\Delta_0)$ consists exactly of the elements b_0 such that $\{\alpha_1\} \cup b_0 \in B$. We conclude by applying the induction hypothesis to the vector space H_0 and the system $\Delta \cap H_0$, for all H_0 .

We see that giving an ordering of Δ , the set B is characterized as the unique generating family (ϕ_b) with $\sum_{b \in B} |b|$ minimum.

Following [7], we now construct the dual basis $(\phi^b)_{b \in B}$ of the basis (ϕ_b) , by using iterated residues. In our framework, they can be introduced as follows.

Let $\alpha \in \Delta$ and let $\Delta \setminus k\alpha$ be the complement in Δ of the set of scalar multiples of α . Denote by Δ/α the image of $\Delta \setminus k\alpha$ in the quotient space $V/k\alpha$. Any $\phi \in S_\Delta$ has at worst a simple pole along $\alpha = 0$. Thus, restriction of $\alpha\phi$ to $(\alpha = 0) = (V/k\alpha)^*$ is a well defined element of $(\Delta/\alpha)^{-1}S(V/k\alpha)$; we denote it by $Res_\alpha(\phi)$. This defines a linear map

$$Res_\alpha : S_\Delta = G_\Delta[-r] \rightarrow G_{\Delta/\alpha}[-r+1] = S_{\Delta/\alpha}.$$

Given an ordered basis $(\beta_1, \dots, \beta_r)$ of Δ , we can iterate this construction to obtain a linear form

$$Res_{\beta_1} Res_{\beta_2} \cdots Res_{\beta_r}$$

on S_Δ . On the other hand, we have a complete flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

(where each V_j is spanned by $\beta_{r-j+1}, \beta_{r-j+2}, \dots, \beta_r$) together with a non-zero element of each $\wedge^j V_j$. Thus, we obtain another linear form

$$Res_{V_1/V_0} \cdots Res_{V_{r-1}/V_{r-2}} Res_{V/V_{r-1}}$$

on S_Δ identified with $S_\Delta \otimes \wedge^r V$.

Proposition 14 1) (Orlik-Solomon-Terao) For any $\alpha \in \Delta$, the map Res_α defines an exact sequence

$$0 \rightarrow S_{\Delta \setminus k\alpha} \rightarrow S_\Delta \rightarrow S_{\Delta/\alpha} \rightarrow 0.$$

2) For any ordered basis $b = (\beta_1, \dots, \beta_r)$ of Δ , we have

$$Res_{\beta_1} Res_{\beta_2} \cdots Res_{\beta_r} = Res_{V_1/V_0} \cdots Res_{V_{r-1}/V_{r-2}} Res_{V/V_{r-1}}$$

with notation as above.

3) (Szenes) The dual basis of $(\phi_b)_{b \in B}$ is given by

$$\phi^b = Res_{\beta_1} Res_{\beta_2} \cdots Res_{\beta_r}.$$

Proof. 1) is a consequence of [6] Theorem 3.126; a direct proof is as follows. Let σ be a basis of Δ containing α , and let σ/α be its image in $V/k\alpha$. Then σ/α is a basis of Δ/α and all bases of Δ/α are obtained in this way. Moreover, $Res_\alpha(\phi_\sigma)$ is a non-zero multiple of $\phi_{\sigma/\alpha}$. It follows that Res_α is surjective.

Clearly, the kernel of Res_α contains $S_{\Delta \setminus k\alpha}$. Conversely, if $\phi \in S_\Delta$ is mapped to 0 by Res_α , then ϕ is defined on $(\alpha = 0)$ and thus, $\phi \in R_{\Delta \setminus k\alpha}$. We can write $\phi = \phi_1 + \phi_2$ where $\phi_1 \in S_{\Delta \setminus k\alpha}$ and $\phi_2 \in V^* R_{\Delta \setminus k\alpha}$. Then $\phi_2 = \phi - \phi_1$ is in S_Δ , too, whence $\phi_2 = 0$ and $\phi \in S_{\Delta \setminus k\alpha}$.

2) Let $b' = (\beta'_1, \dots, \beta'_r)$ be another ordered basis of Δ . Consider the element

$$\rho := Res_{\beta_1} \cdots Res_{\beta_r}(\phi_{b'}).$$

If $Res_{\beta_r}(\phi_{b'})$ is non-zero, then we must have $\beta_r = t_r \beta'_{\pi(r)}$ for some non-zero $t_r \in k$ and some index $\pi(r)$. If moreover $Res_{\beta_{r-1}} Res_{\beta_r}(\phi_{b'})$ is non-zero, then we must have $\beta_{r-1} \in t_{r-1} \beta'_{\pi(r-1)} + k \beta'_{\pi(r)}$ for some non-zero $t_{r-1} \in k$ and some $\pi(r-1) \neq \pi(r)$ (because β_{r-1} is not a multiple of β_r). Continuing in this way, we see that either $\rho = 0$ or there exists a permutation π of $\{1, \dots, r\}$ and non-zero $t_1, \dots, t_r \in k$ such that

$$\beta_i \in t_i \beta'_{\pi(i)} + \sum_{j=i+1}^r k \beta'_{\pi(j)}$$

for all i . Then we have $\rho = t_1 \cdots t_r$.

On the other hand, set

$$\rho' := Res_{V_1/V_0} \cdots Res_{V/V_{r-1}}(\phi_{b'}).$$

If $\text{Res}_{V/V_{r-1}}(\phi_{b'}) \neq 0$, then there exist a unique index $\pi'(1)$ and a unique non-zero $t'_1 \in k$ such that $\beta'_{\pi'(1)} \in t'_1 \beta_1 + \sum_{j=2}^r k \beta_j$. Further, $\beta'_i \in \sum_{j=2}^r k \beta_j$ for all $i \neq \pi'(1)$. If moreover $\text{Res}_{V_{r-1}/V_{r-2}} \text{Res}_{V/V_{r-1}}(\phi_{b'}) \neq 0$, then $\beta'_{\pi'(r-1)} \in t'_2 \beta_2 + \sum_{j=3}^r k \beta_j$ for uniquely defined $\pi'(r-1)$ and t'_{r-1} . Further, $\beta'_i \in \sum_{j=3}^r k \beta_j$ for all $i \notin \{\pi'(1), \pi'(2)\}$. Continuing, we obtain if $\rho' \neq 0$:

$$\beta'_{\pi'(i)} \in t'_i \beta_i + \sum_{j=i+1}^r k \beta_j$$

for a permutation π' and non-zero t'_1, \dots, t'_r ; then we have $\rho' = 1/t'_1 \cdots t'_r$. This is equivalent to the set of conditions of the first part of the proof, with $\pi' = \pi$ and $t'_i = 1/t_i$.

3) Let $b' = (\beta'_1, \dots, \beta'_r) \in B$ such that $\text{Res}_{\beta_1} \cdots \text{Res}_{\beta_r}(\phi_{b'}) \neq 0$. Let π and t_1, \dots, t_r be as above; then β_{r-1} , $\beta'_{\pi(r-1)}$ and $\beta_r = t_r \beta'_{\pi(r)}$ are linearly dependent. Write $\beta_{r-1} = \alpha_i$, $\beta'_{\pi(r-1)} = \alpha_{i'}$ and $\beta_r = \alpha_j$, then $j > i$ and $j > i'$. If $i > i'$ then $\alpha_{i'}$, β_{r-1} and β_r are linearly dependent, which contradicts the hypothesis $b \in B$. Similarly, we cannot have $i < i'$. Thus, $i = i'$, that is, $\beta_{r-1} = \beta'_{\pi(r-1)}$. In this way we obtain $\beta_q = \beta'_{\pi(q)}$ for all q ; because b and b' are in B , it follows that $b = b'$.

5 Laplace transform and Jeffrey-Kirwan residue

Starting from now, we assume that $k = \mathbb{R}$. The Laplace transform associates to a polynomial function supported on an acute cone in V a rational function on V^* . We will define the inverse Laplace transform, formally denoted by $\int_{V^*}^\delta e^{\langle y, h \rangle} [[\phi(y)]] dy$, of a meromorphic function ϕ on V^* with poles on a set of hyperplanes. It depends of a choice of a chamber δ in V^* and is a locally polynomial function on the dual cone δ^\vee of δ .

Let V be an oriented real vector space of dimension r . We denote by o its orientation. We consider as before a finite subset Δ of $V \setminus \{0\}$, which spans V ; we assume moreover that $-\alpha \in \Delta$ for all $\alpha \in \Delta$. A wall of V is defined to be an hyperplane generated by $r-1$ linearly independent elements of Δ . We denote by $\mathcal{H}(\Delta)$ the union of walls. This is a set of hyperplanes in V . The set

$$V_{\text{reg}, \Delta} := V - \mathcal{H}(\Delta)$$

is the set of regular elements in V .

We define the vector space \mathcal{P}_Δ of locally polynomial functions on $V_{reg,\Delta}$. Elements of \mathcal{P}_Δ are given by polynomial functions on each connected component of $V_{reg,\Delta}$. The space $S(V^*)$ identifies to the space of polynomial functions on V . Thus the space \mathcal{P}_Δ is a module under the action of $S(V^*)$ by multiplication.

Let $C \subset V$ be an acute convex cone with non empty interior. Let $C^\vee \subset V^*$ be its (closed) dual cone. Then the interior of C^\vee is not empty. We denote by $[C]$ the characteristic function of C , that is, the function with value 1 on C and 0 outside C .

Let $f \in S(V^*)$ be a polynomial function on V . Let dh be an element of $\wedge^r V^*$. As V is oriented, we can integrate over V a differential form $\alpha = \phi(h)dh$ of maximal degree (here ϕ is an integrable function on V). We denote the integral over V of such a differential form α by $\int_{V,o} \alpha$. A change of orientation produces a change of sign.

For each y in the interior of C^\vee , the integral

$$L^o(f[C])(y) := \int_{V,o} e^{-\langle y,h \rangle} f(h)[C](h)dh$$

converges, and defines the *Laplace transform* of $f[C]$. If C is generated by multiples of elements of Δ , it is easy to see that $L^o(f[C])$ is given by the restriction to the interior of C^\vee of a rational function on V^* belonging to the subspace G_Δ of R_Δ . We still denote this rational function by $L^o(f[C])(y)$. More exactly, as $L^o(f[C])(y)$ depends linearly on dh , we see that $L^o(f[C])(y)$ is a rational function with values in $(\wedge^r V^*)^* = \wedge^r V$. Furthermore it is clear that the map L^o interchanges the action of $S(V^*)$ by multiplication on \mathcal{P}_Δ with its action by derivation on G_Δ , up to the automorphism $P(h) \mapsto P(-h)$.

Let δ be a connected component of the set $V^* - \mathcal{H}^*(\Delta)$. Then δ is an open acute polyhedral cone in V^* , and δ^\vee is a closed acute polyhedral cone in V . We denote by $\mathcal{P}_\Delta(\delta)$ the subspace of \mathcal{P}_Δ spanned by functions $f(h)[C(\sigma)](h)$ where $f \in S(V^*)$ and σ is a basis of Δ **such that** $C(\sigma) \subset \delta^\vee$; here $C(\sigma)$ denotes the closed convex cone generated by σ .

If dh is a positive element of $\wedge^{max} V^*$, we denote by $vol(\sigma, dh)$ the volume of the parallelepiped constructed on the basis σ for the positive density dh corresponding to the differential form dh . Specifically, if $dh = e^1 \wedge e^2 \wedge \dots \wedge e^r$ and if $\sigma = \{\alpha_1, \dots, \alpha_r\}$, we have $vol(\sigma, dh) = |\det \langle \alpha_i, e^j \rangle_{i,j}|$. Finally, we denote by L_δ^o the restriction to $\mathcal{P}_\Delta(\delta)$ of the Laplace transform L^o .

Theorem 15 *Given any chamber δ in V^* , the Laplace transform*

$$L_\delta^o : \mathcal{P}_\Delta(\delta) \rightarrow G_\Delta \otimes \wedge^{\max} V$$

is an isomorphism and commutes with the actions of $S(V^)$ up to the automorphism $P(h) \mapsto P(-h)$.*

We have, for $dh \in \wedge^{\max} V^$ positive (with respect to our choice of orientation o), and σ a basis of Δ such that $C(\sigma) \subset \delta^\vee$,*

$$\langle L_\delta^o[C(\sigma)], dh \rangle = \text{vol}(\sigma, dh) \phi_\sigma$$

that is, for $y \in \delta$, we have the equality of functions

$$\int_{C(\sigma), o} e^{-\langle y, h \rangle} dh = \text{vol}(\sigma, dh) \phi_\sigma(y).$$

Proof. The formula for the Laplace transform of $[C(\sigma)]$ is straightforward. It implies surjectivity of L_δ^o because this map is $S(V^*)$ -linear, and the $S(V^*)$ -module G_Δ is generated by the ϕ_σ where σ is a basis of Δ such that $C(\sigma)$ is contained in δ^\vee (here we use the assumption that Δ is centrally symmetric).

For injectivity of L_δ^o , we observe that any function $\phi \in \mathcal{P}_\Delta(\delta)$ is supported in the acute cone δ^\vee , and that $\phi = 0$ if and only if ϕ vanishes outside a set of measure zero. Moreover, the set of functions $h \mapsto e^{-\langle y, h \rangle}$ (where $y \in \delta$) is dense in the space of smooth, rapidly decreasing functions on δ^\vee .

Consider the inverse $(L_\delta^o)^{-1} : G_\Delta \otimes \wedge^{\max} V \mapsto \mathcal{P}_\Delta(\delta)$. Via the projection map Princ_Δ , we can extend the map $(L_\delta^o)^{-1}$ to $\Delta^{-1} \hat{S}(V) \otimes \wedge^{\max} V$. Thus we set

$$F_\delta^o(\phi \otimes dy) := (L_\delta^o)^{-1}(\text{Princ}_\Delta(\phi) \otimes dy).$$

Thus, F_δ^o associates to any meromorphic function ϕ on V^* with poles on the hyperplanes $\alpha = 0$ a locally polynomial function on δ^\vee . We denote $F_\delta^o(\phi \otimes dy)$ by the formal notation:

$$(F_\delta^o(\phi \otimes dy))(h) = \int_{V^*, o}^\delta e^{\langle y, h \rangle} [[\phi(y)]] dy.$$

We now show that F_δ^o commutes with the actions of $S(V)$ by derivations on $\mathcal{P}_\Delta(\delta)$, and by multiplication on R_Δ .

Lemma 16 *For any $\psi \in S(V)$ and $\phi \in G_\Delta$, we have*

$$\psi(\partial)F_\delta^o(\phi \otimes dy) = F_\delta^o(\psi\phi \otimes dy).$$

Proof. It is enough to check this for $\psi = v \in V$. Then, for any $y \in \delta$, we have

$$\begin{aligned} & \int_V (\partial(v)F_\delta^o(\phi))(h)e^{-\langle y, h \rangle} dh - \langle v, y \rangle \phi(y) \\ &= \int_V (\partial(v)F_\delta^o(\phi))(h)e^{-\langle y, h \rangle} dh + \int_V F_\delta^o(\phi)(h)\partial(v)(e^{-\langle y, h \rangle})dh \\ &= \int_V \partial(v)(F_\delta^o(\phi)(h)e^{-\langle y, h \rangle})dh = \int_\Sigma F_\delta^o(\phi)(h)e^{-\langle y, h \rangle} i_v(dh) \end{aligned}$$

where Σ denotes the boundary of the support of $F_\delta^o(\phi)$; here the latter equality follows from Stokes' theorem. Because Σ is a union of polyhedral cones of smaller dimensions, the function

$$y \mapsto \int_\Sigma F_\delta^o(\phi)(h)e^{-\langle y, h \rangle} i_v(dh)$$

is in NG_Δ . We thus have

$$L_\delta^o(\partial(v)F_\delta^o(\phi \otimes dy)) - v\phi \in NG_\Delta$$

which implies our formula.

If $\phi \in S_\Delta$, the image $F_\delta^o(\phi \otimes dy)$ is a locally constant function on $V_{reg, \Delta}$. Thus we obtain a number of residue maps defined by chambers γ in V and δ in V^* :

$$Res_{\gamma, \delta} : S_\Delta \otimes \wedge^{max} V \rightarrow \mathbf{R},$$

$$\phi \otimes dy \mapsto (F_\delta^o \phi)|_\gamma.$$

The formula of Theorem 15 determines $Res_{\gamma, \delta}(\phi_\sigma \otimes dy)$ for $C(\sigma) \subset \delta^\vee$ and dy a positive element of $\wedge^r V$. More precisely, if dh is the dual measure to dy ,

$$Res_{\gamma, \delta}(\phi_\sigma \otimes dy) = \frac{1}{\text{vol}(\sigma, dh)}, \quad \text{if } \gamma \subset C(\sigma),$$

$$Res_{\gamma, \delta}(\phi_\sigma \otimes dy) = 0, \quad \text{if } \gamma \cap C(\sigma) = \emptyset.$$

As F_δ^o commutes with the action of differential operators with constant coefficients, we have

$$F_\delta^o(P(\partial)\phi_\sigma \otimes dy) = P(-h)F_\delta^o(\phi_\sigma \otimes dy)(h)$$

so that if $C(\sigma) \subset \delta^\vee$

$$(1) \quad F_\delta^o(P(\partial)\phi_\sigma \otimes dy)(h) = \frac{1}{\text{vol}(\sigma, dh)} P(-h)[C(\sigma)](h).$$

Proposition 17 (*Jeffrey-Kirwan*) For $\phi \in \hat{R}_\Delta$ and $h \in V$, we have

$$F_\delta^o(\phi \otimes dy)(h) = F_\delta^o(\text{Res}_\Delta(e^h \phi) \otimes dy).$$

Proof. It is sufficient to prove this formula for $\phi = P(\partial)\phi_\sigma$. As we have for $y \in V^*$:

$$\text{Res}_\Delta(e^h \partial(y)\phi) = -\text{Res}_\Delta((\partial(y)e^h)\phi) = -\langle y, h \rangle \text{Res}_\Delta(e^h \phi),$$

we obtain

$$\text{Res}_\Delta(e^h P(\partial)\phi_\sigma) = P(-h)\text{Res}_\Delta(e^h \phi_\sigma) = P(-h)\phi_\sigma.$$

So we see, from Formula (1) above, that the equation of Proposition 17 is satisfied.

Proposition 17 provides an effective tool to compute the inverse Laplace transform of a rational function ϕ with poles on hyperplanes. Indeed, the function $\text{Res}_\Delta(e^h \phi)$ is an element of S_Δ (depending of h), so that it can be written as a linear combination

$$\text{Res}_\Delta(e^h \phi) = \sum_{\sigma} c_{\sigma}(h) \phi_{\sigma}.$$

The choice of a chamber δ determines a sign $\epsilon(\sigma, \delta)$ for which

$$\phi_{\sigma} = \epsilon(\sigma, \delta) \phi_{\sigma^\delta}$$

where the cone σ^δ has the same axes as $C(\sigma)$ and is contained in δ^\vee . Thus, the restriction to a chamber γ in V of the inverse Laplace transform $F_\delta^o(\phi \otimes dy)$

is obtained by summing the polynomial terms $\epsilon(\sigma, \delta)c_\sigma(h) \text{vol}(\sigma, dh)^{-1}$ for all σ such that $\gamma \subset C(\sigma^\delta)$:

$$F_\delta^o(\phi \otimes dy)|_\gamma = \sum_{\sigma, \gamma \subset C(\sigma^\delta)} \epsilon(\sigma, \delta)c_\sigma(h) \text{vol}(\sigma, dh)^{-1}.$$

This is Jeffrey-Kirwan algebraic formula.

Example

Let us consider a two-dimensional vector space V with basis (e_1, e_2) . Let $\Delta = \{e_1, e_2, e_1 + e_2\}$. Consider

$$\phi(z_1, z_2) = \frac{1}{z_1 z_2 (z_1 + z_2)}.$$

We have

$$\begin{aligned} \text{Res}_\Delta(e^{h_1 z_1 + h_2 z_2} \phi(z_1, z_2) \otimes dz_1 dz_2) &= \frac{h_1 z_1 + h_2 z_2}{z_1 z_2 (z_1 + z_2)} \\ &= \frac{h_1}{z_2 (z_1 + z_2)} + \frac{h_2}{z_1 (z_1 + z_2)}. \end{aligned}$$

If δ is the component $e_1 > 0, e_2 > 0$ of V^* , we then obtain the following picture for the inverse Laplace transform of $\frac{1}{z_1 z_2 (z_1 + z_2)}$.

In the next section, we determine the change of $F_\delta^o \phi$ when jumping over a wall.

6 The jump formula

We consider, as in Section 5, a real oriented vector space (V, o) with a system of hyperplanes defined by $\Delta \subset V - \{0\}$. Let δ be a chamber in V^* , and let F_δ^o be the inverse Laplace transform. In this section, we relate the jumps of $F_\delta^o(\phi) \otimes dy$ across walls, with the poles of the function ϕ along the wall.

Let (V_0, o_0) be an oriented wall with its system $\Delta_0 = \Delta \cap V_0$. The wall V_0 separates V in two half-spaces. Choose an equation z of V_0 such that $o = z \wedge o_0$, and define

$$V_+ = \{h \in V, \langle z, h \rangle > 0\},$$

$$V_- = \{h \in V, \langle z, h \rangle < 0\}.$$

If U is a component of $(V_0)_{reg, \Delta_0}$ there exists unique components U_{\pm} of $V_{reg, \Delta}$ contained in V_{\pm} and such that $U \subset \overline{U_{\pm}}$.

Let $f \in \mathcal{P}_{\Delta}$ be a locally polynomial function on $V_{reg, \Delta}$. Then the restriction of f to U_+ (resp. U_-) is given by a polynomial function f^+ (resp. f^-). We define the locally polynomial function $Jump_{o/o_0}(f) \in \mathcal{P}_{\Delta_0}$ by the formula

$$Jump_{o/o_0}(f)|_U = f^+|_U - f^-|_U.$$

Theorem 18 *Let (V_0, o_0) be an oriented wall. Let δ be a chamber in V^* and δ_0 a chamber in V_0^* such that $\delta_0^{\vee} \subset \delta^{\vee}$. Then, for any $\phi \in \hat{R}_{\Delta}$, we have the Jump formula:*

$$Jump_{o/o_0}(F_{\delta}^o(\phi \otimes dy_0)) = F_{\delta_0}^{o_0}(Res_{V/V_0}(\phi \otimes dy)).$$

Proof. It is sufficient to prove this formula for $\phi \in G_{\Delta}$. (On NG_{Δ} , both sides are equal to 0, because Res_{V/V_0} maps NG_{Δ} to NG_{Δ_0}). Thus it is sufficient to prove this formula for a derivative $\phi = P(\partial)\phi_{\sigma}$ of an element ϕ_{σ} , with $C(\sigma) \subset \delta^{\vee}$. Then

$$F_{\delta}^o(\phi \otimes dy)(h) = P(-h)[C(\sigma)](h).$$

If V_0 is not a wall of $C(\sigma)$, then $F_{\delta}^o(\phi \otimes dy)$ has no jump along V_0 . Thus the left-hand side of the equality in Theorem 18 is equal to 0. The right-hand side is also 0, as there are at least 2 vectors in σ which are not in Δ_0 .

If V_0 is a wall of $C(\sigma)$, there exists $\beta \in \Delta$ such that $\sigma = \sigma_0 \cup \{\beta\}$ where σ_0 is a basis of V_0 . Write $V = V_0 \oplus \mathbb{R}\beta$. Write an element $h \in V$ as $h = h_0 + h_1\beta$ with $h_0 \in V_0$ and $h_1 \in \mathbb{R}$. Then the left-hand side is the function $P(-h_0)[C(\sigma_0)](h_0)$. If P is divisible by h_1 , then $Res_{V/V_0}(P(\partial)\phi_{\sigma} \otimes dy) = 0$. Thus, both sides vanish. If P only depends on h_0 , then

$$Res_{V/V_0}(P(\partial)\phi_{\sigma} \otimes dy) = P(\partial)\phi_{\sigma_0} \otimes dy_0$$

whence the right-hand side is $P(-h_0)[C(\sigma_0)](h_0)$.

As an application of the Jump formula, let us relate the behaviour at infinity of a function $\phi \in G_{\Delta}$ to the order of differentiability of its inverse Laplace transform. For a positive integer n , we say that ϕ **vanishes at order n at infinity** if the rational function $t \mapsto t^{n-1}\phi(y + tz)$ is 0 at ∞ for all regular $y \in V^*$ and for all $z \in V^*$. Equivalently, $\psi\phi \in G_{\Delta}$ for any $\psi \in S(V)$ of degree at most $n - 1$ (indeed, recall that G_{Δ} is the space of functions that vanish at infinity).

Corollary 19 *For a function $\phi \in G_\Delta$ and a non-negative integer k , the following conditions are equivalent:*

- 1) $F_\delta^o(\phi \otimes dy)$ extends to a function of class C^k on V .
- 2) ϕ vanishes at order $k + 2$ at infinity.

Further, for a wall V_0 with equation $z_0 = 0$ and for ϕ satisfying (1) or (2), the following conditions are equivalent:

- 1)' $F_\delta^0(\phi \otimes dy)$ extends to a function of class C^{k+1} along V_0 .
- 2)' For any regular $z \in V^*$, the rational function $t \mapsto \phi(z + tz_0)$ vanishes at order $k + 3$ at infinity.

Proof. Observe that $F_\delta^o(\phi \otimes dy)$ extends to a continuous function on V if and only if it has no jumps along walls. This amounts to $\text{Res}_{V/V_0}(\phi \otimes dy) = 0$ for any wall V_0 (because Res_{V/V_0} maps $G_\Delta \otimes \wedge^r V$ to $G_{\Delta_0} \otimes \wedge^{r-1} V_0$, and $F_{\delta_0}^{o_0}$ is injective on the latter). Equivalently,

$$\text{Res}_{t=\infty}(\phi(z + tz_0)dt) = 0$$

for all regular z and for all z_0 . Because ϕ vanishes at infinity, this means that ϕ vanishes at order 2 there. This proves the equivalence of (1) and (2) in the case where $k = 0$.

The general case follows by induction on k . Indeed, recall that

$$\partial(v)F_\delta^o(\phi \otimes dy) = F_\delta^o(v\phi \otimes dy) = F_\delta^o(\text{Princ}_\Delta(v\phi) \otimes dy)$$

for any $v \in V$. Thus, using the induction hypothesis for $k - 1$, assertion (1) is equivalent to: ϕ and $\text{Princ}_\Delta(v\phi)$ vanish at order $k + 1$ at infinity. Then $v\phi \in G_\Delta$ (because ϕ vanishes at order 2 at infinity) and (1) is equivalent to: $v\phi$ vanishes at order $k + 1$ at infinity.

The proof of equivalence of (1)' and (2)' is similar.

7 Orlik-Solomon relations and stratified Fourier transform

We still consider a real vector space V with a finite subset $\Delta \subset V \setminus \{0\}$ such that Δ spans V and $\Delta = -\Delta$. We fix a Lebesgue measure dh on V and a chamber $\delta \subset V^*$. Changing slightly notation, the inverse Laplace transform F_δ associates to any element of G_Δ a locally polynomial function on $V_{\text{reg}, \Delta}$. In this section, we associate to any element of G_Δ a **piecewise polynomial**

function defined on all of V . This assignement will depend on the choices of a chamber δ in V^* and of a chamber γ in V ; it will be denoted by $F_{\gamma,\delta}$. The piecewise polynomial function $F_{\gamma,\delta}(\phi)$ will extend the locally polynomial function $F_\delta(\phi)$, and will be the continuous extension of $F_\delta(\phi)$ if it exists. We will use the function $F_{\gamma,\delta}(\phi)$ in part III of this article, in connection with the definition of Eisenstein series.

Denote by \mathcal{PP}_Δ the vector space of functions on V spanned by functions $P[C]$ where $P \in S(V^*)$ and C is a closed polyhedral cone with axes in Δ . Then \mathcal{PP}_Δ is a $S(V^*)$ -submodule of the module of piecewise polynomial functions on V (for the stratification where the open strata are the chambers, and the closures of other strata are proper faces of closures of chambers). We begin by constructing a morphism of $S(V^*)$ -modules from G_Δ to a quotient of \mathcal{PP}_Δ . This morphism will depend on the choice of a chamber γ in V , will be denoted by F_γ , and will be called the formal Fourier transform.

For a basis σ of V , we denote by $|\det(\sigma)|$ the volume of the parallelepiped constructed on σ . We set

$$a_\sigma := |\det(\sigma)|\phi_\sigma = \frac{|\det(\sigma)|}{\prod_{\alpha \in \sigma} \alpha},$$

an element of S_Δ . Remark that a_σ does not change if we multiply elements in σ by positive constants. The Orlik-Solomon relations are more naturally expressed in terms of the a_σ , as shown by the following result, an easy consequence of Theorem 1 and Proposition 13.

Proposition 20 *Let σ be a basis of Δ , let $\alpha \in \Delta \setminus \sigma$ and let $\alpha = \sum_{\beta \in \sigma} c_{\alpha\beta}\beta$ be the expansion of α in the basis σ . Then the elements $(a_\sigma)_{\sigma \in \mathcal{B}(\Delta)}$ verify the relations*

$$(OS) \quad a_\sigma = \sum_{\beta \in \sigma, c_{\alpha\beta} \neq 0} \text{sign}(c_{\alpha\beta}) a_{\sigma \cup \{\alpha\} \setminus \{\beta\}}.$$

Furthermore, if \mathcal{M} is a $S(V^*)$ -module and $(A_\sigma)_{\sigma \in \mathcal{B}(\Delta)}$ is a family in \mathcal{M} verifying the relations (OS), then there exists a unique map $A : R_\Delta \rightarrow \mathcal{M}$ such that

- 1) the map A commutes with the action of $S(V^*)$.
- 2) For all $\sigma \in \mathcal{B}(\Delta)$, we have $A(a_\sigma) = A_\sigma$.
- 3) $A(NG_\Delta) = 0$.

Remark that the relations (OS) have coefficients equal to ± 1 . It makes thus sense to find elements in an abelian group, satisfying these relations. The group $\mathcal{C}(V)$ generated by characteristic functions of locally closed polyhedral cones in V will be very useful to construct such elements A_σ .

We introduce some notation. A *polyhedral cone* in V is a closed convex cone $C \subset V$ (with vertex at 0) which is generated by finitely many vectors h_1, \dots, h_n ; we set $C = C(h_1, \dots, h_n)$. For A a subset of V , we denote by $[A]$ the characteristic function of A , i.e., the function on V with value 1 on A and 0 outside A . We denote by $\mathcal{C}(V)$ the additive group of integral valued functions on V , generated by all characteristic functions of polyhedral cones.

For any closed convex cone C , we denote by C^0 the relative interior of C , i.e., the interior of C in the affine space generated by C . Observe that $\mathcal{C}(V)$ contains the characteristic functions of relative interiors of polyhedral cones, and more generally, the characteristic functions of locally closed polyhedral cones. The subgroup of $\mathcal{C}(V)$ generated by characteristic functions of polyhedral cones which contain lines is denoted by $\mathcal{LC}(V)$. For example if $\alpha \in V$ is nonzero, then

$$[C(-\alpha)] + [C(\alpha)^0] \in \mathcal{LC}(V).$$

We denote by \mathcal{C}_Δ the subspace of $\mathcal{C}(V)$ generated by characteristic functions of polyhedral cones $C(\kappa)$ where $\kappa \subset \Delta$. Then, by definition, \mathcal{PP}_Δ is the $S(V^*)$ -module generated by \mathcal{C}_Δ . We denote by \mathcal{LC}_Δ the subspace of \mathcal{C}_Δ generated by functions $[C(\kappa)]$ where $\kappa \subset \Delta$ and $C(\kappa)$ contains a line.

Let $p \in V$ and let $C \subset V$ be a polyhedral cone with non-empty interior, such that p lies in no hyperplane generated by a facet of C . Set

$$C'_p := \{h \in C \mid \text{the segment } [h, p] \text{ meets } C^0\}.$$

Then C'_p is equal to C minus the union of its facets which generate a hyperplane separating C^0 and p . In particular, C'_p is a locally closed polyhedral cone. If moreover $C = C(\sigma)$ where $\sigma \in \mathcal{B}(\Delta)$, and $p = \sum_{\alpha \in \sigma} p_\alpha \alpha$ is in $V_{reg, \Delta}$, then we obtain easily

$$C(\sigma)'_p := C(\alpha, p_\alpha > 0) + C(\alpha, p_\alpha < 0)^0.$$

In particular, the cone $C(\sigma)'_p$ depends only of the chamber γ which contains p . Thus, we denote it by $C(\sigma)'_\gamma$.

Define a map

$$A_\gamma : \mathcal{B}(\Delta) \rightarrow \mathcal{C}_\Delta$$

by

$$A_\gamma(\sigma) = [C(\sigma)'_\gamma].$$

Theorem 21 *For any chamber γ in V , the family of elements $(A_\gamma(\sigma))_{\sigma \in \mathcal{B}(\Delta)}$ verify the relations (OS) in the quotient group $\mathcal{C}_\Delta / \mathcal{LC}_\Delta$.*

Proof. Because of the relation $[C(-\alpha)] = -[C(\alpha)^0]$ modulo \mathcal{LC}_Δ we see that the image of the element $A_p(\sigma)$ in $\mathcal{C}_\Delta / \mathcal{LC}_\Delta$ changes sign, if we flip one of the elements β_j in $\sigma = (\beta_1, \beta_2, \dots, \beta_r)$ to $-\beta_j$. We thus may assume that the relation is

$$\alpha = \beta_1 + \beta_2 + \dots + \beta_s$$

for some $s \leq r$. Then the cones $C(\sigma \cup \{\alpha\} \setminus \{\beta_j\})$ ($1 \leq j \leq s$) are the maximal cones in a polyhedral subdivision of $C(\sigma)$, and we conclude by the lemma below.

Lemma 22 *Let $C \subset V$ be a polyhedral cone. Let C_1, \dots, C_n be the maximal cones of a polyhedral subdivision of C . Let $p \in V$ such that p lies in no hyperplane generated by a facet of some C_i . Then C'_p is the disjoint union of $C'_{1,p}, \dots, C'_{n,p}$.*

Proof. Clearly, each $C'_{i,p}$ is contained in C'_p . Conversely, let $x \in C'_p$. If x lies in no $C'_{i,p}$ then the segment $[x, p] \cap C^0$ has a non-empty interior in $[x, p]$ and is contained in the union of all facets of the C_i . It follows that this segment is contained in a facet of some C_i . Thus, p is in the hyperplane generated by this facet, a contradiction. So $x \in C'_{i,p}$ for some i . Assume that $x \in C'_{j,p}$ for some $j \neq i$. Then $[x, p] \cap C_i^0$ and $[x, p] \cap C_j^0$ are disjoint segments with non-empty interiors in $[x, p]$. Moreover, because $x \in C_i \cap C_j$, the closures of both segments contain x , a contradiction.

We denote by \mathcal{LP}_Δ the $S(V^*)$ -submodule of \mathcal{PP}_Δ generated by \mathcal{LC}_Δ , that is, the space of piecewise polynomial functions which are polynomial in at least one direction. By the preceding theorem, together with Proposition 20, each choice of a chamber γ in V defines a morphism of $S(V^*)$ -modules F_γ from G_Δ to the quotient space $\mathcal{PP}_\Delta / \mathcal{LP}_\Delta$.

Definition 23 *Let γ be a connected component of $V_{reg, \Delta}$. We denote by*

$$F_\gamma : R_\Delta \rightarrow \mathcal{PP}_\Delta / \mathcal{LP}_\Delta$$

the unique map such that

- 1) F_γ commutes with the action of $S(V^*)$ up to the automorphism $P(h) \mapsto P(-h)$.
- 2) $F_\gamma(|\det(\sigma)|\phi_\sigma) = [C(\sigma)'_\gamma]$ for all $\sigma \in \mathcal{B}(\Delta)$.
- 3) $F_\gamma(NG_\Delta) = 0$.

We call F_γ the formal Fourier transform.

Now we construct a lift of $F_\gamma : R_\Delta \rightarrow \mathcal{PP}_\Delta/\mathcal{LP}_\Delta$ to \mathcal{PP}_Δ . In other words, we associate to any element of R_Δ a piecewise polynomial function on V , compatibly with F_γ . We may do this by **specifying a chamber in V^*** , as shown by

Lemma 24 *Let δ be a chamber in V^* and let $\phi \in R_\Delta$. Then $F_\gamma(\phi)$ has a unique representative with support in δ^\vee .*

Proof. Let $\alpha \in \Delta$, then α or $-\alpha$ is in δ^\vee . Using the relation $[C(-\alpha)] + [C(\alpha)^0] \in \mathcal{LC}_\Delta$, we see that $C(\kappa)$ has a representative with support in δ^\vee , for any linearly independent $\kappa \subset \Delta$. This shows existence. For uniqueness, it is enough to check that any $f \in \mathcal{LP}_\Delta$ with support in some acute cone C must be zero. This is shown in the proof of [1] Theorem 1.4 for $f \in \mathcal{LC}(V)$; this proof adapts with minor changes, as follows. Embed \mathcal{LP}_Δ into the vector space $\mathcal{F}(V^*)$ of functions on V^* . The additive group of V^* acts on $\mathcal{F}(V^*)$ by translations; we denote by $z \mapsto T(z)$ this action. For a polyhedral cone C which contains a line l , we have $(1 - T(z))[C] = 0$ for all $z \in l$. Thus, for $P \in S(V^*)$, we have

$$(1 - T(z))^N(P[C]) = 0$$

whenever $N > \deg(P)$. Because $f \in \mathcal{LP}_\Delta$, it follows that there exist $z_1, \dots, z_n \in V^* \setminus \{0\}$ (non necessarily distinct) such that

$$\prod_{j=1}^n (1 - T(t_j z_j))f = 0$$

for all $t_j \in \mathbb{R}$. Moreover, we can find $h \in V$ such that $h > 0$ on $C \setminus \{0\}$ and that $\langle h, z_j \rangle \neq 0$ for all j . Replacing z_j by $-z_j$, we may assume that $\langle h, z_j \rangle < 0$ for all j . Let $w \in V^*$. We can choose $A > 0$ such that

$$\langle h, w + \sum_{j \in J} t_j z_j \rangle < 0$$

for any non-empty subset J of $\{1, \dots, n\}$ and for $t_j > A$. We have

$$0 = \left(\prod_{j=1}^n (1 - T(-t_j z_j)) f \right)(w) = \sum_{J \subset \{1, \dots, n\}} (-1)^{|J|} f(w + \sum_{j \in J} t_j z_j).$$

By assumption, f is identically zero on the open half-space $h < 0$. It follows that $f(w) = 0$.

We denote by $F_{\gamma, \delta}(\phi)$ the representative of $F_{\gamma}(\phi)$ with support in γ^{\vee} . Let us compute $F_{\gamma, \delta}(a_{\sigma})$ for $\sigma \in \mathcal{B}(\Delta)$. Write

$$a_{\sigma} = \epsilon(\sigma, \delta) a_{\sigma^{\delta}}$$

where $\epsilon(\sigma, \delta) = \pm 1$ and the cone $C(\sigma^{\delta})$ has the same axes as $C(\sigma)$ and is contained in δ^{\vee} . Then, by definition

$$F_{\gamma, \delta}(a_{\sigma}) = \epsilon(\sigma, \delta) [C(\sigma^{\delta})]_{\gamma}'.$$

These elements $F_{\gamma, \delta}(a_{\sigma})$ satisfy the Orlik-Solomon relations in the space \mathcal{PP}_{Δ} .

Definition 25 *Let γ be a chamber in V and let δ be a chamber in V^* . We denote by*

$$F_{\gamma, \delta} : R_{\Delta} \rightarrow \mathcal{PP}_{\Delta}$$

the unique map such that

1) $F_{\gamma, \delta}$ commutes with the action of $S(V^*)$ up to the automorphism $P(h) \mapsto P(-h)$.

2) $F_{\gamma, \delta}(|\det(\sigma)|\phi_{\sigma}) = [C(\sigma)']_{\gamma}$ for all $\sigma \in \mathcal{B}(\Delta)$ such that $C(\sigma) \subset \delta^{\vee}$.

3) $F_{\gamma, \delta}(NG_{\Delta}) = 0$.

We call $F_{\gamma, \delta}$ the stratified Fourier transform.

We now express $F_{\gamma, \delta}(\phi)$ in terms of $F_{\delta}(\phi)$.

Proposition 26 *Let γ be a chamber of V , let $p \in \gamma$ and let δ be a chamber in V^* . Then we have for any $\phi \in G_{\Delta}$ and $h \in V$:*

$$F_{\gamma, \delta}(\phi)(h) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} F_{\delta}(\phi)(h + \epsilon p).$$

In particular, $F_{\gamma, \delta}(\phi)$ is an extension of $F_{\delta}(\phi)$ to the whole of V , and is the continuous extension if it exists.

Proof. Observe first that the formula makes sense: because p is regular, $h + \epsilon p$ is regular for ϵ sufficiently small and $\epsilon > 0$. If the formula holds for ϕ then it holds for $P(\partial)\phi$ where $P \in S(V^*)$, because both F_δ and $F_{\gamma,\delta}$ are $S(V^*)$ -linear. Thus it suffices to check the formula for $\phi = \phi_\sigma$ where $C(\sigma) \subset \gamma^\vee$. Then $F_{\gamma,\delta}(\phi) = [C(\sigma)'_p]$ whereas $F_\delta(\phi)$ is restriction of $[C(\sigma)]$ to $V_{reg,\Delta}$. But

$$[C(\sigma)'_p](h) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} [C(\sigma)](h + \epsilon p)$$

as follows from the definition of C'_p .

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